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Automorphisms and Verma modules for generalized Schrödinger–Virasoro algebras ☆

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ABSTRACT

Let $\mathbb{F}$ be a field of characteristic 0, G an additive subgroup of $\mathbb{F}$, $\alpha \in \mathbb{F}$ satisfying $\alpha \notin G$ and $2\alpha \in G$. We define a class of infinite-dimensional Lie algebras which are called generalized Schrödinger–Virasoro algebras and denoted by $\mathfrak{gsv}[G,\alpha]$. In this paper the automorphism group and irreducibility of Verma modules for $\mathfrak{gsv}[G,\alpha]$ are completely determined.

1. Introduction

The Schrödinger–Virasoro algebra $\mathfrak{sv}$ is defined to be a Lie algebra with $\mathbb{F}$-basis $\{L_n, M_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}$ subject to the following Lie brackets

$$[L_m, L_n] = (n-m)L_{n+m}, \quad [L_m, M_n] = nM_{n+m}, \quad [L_m, Y_{n+\frac{1}{2}}] = \left(n + \frac{1-m}{2}\right)Y_{m+n+\frac{1}{2}},$$
$$[M_m, M_n] = 0, \quad [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (n-m)M_{m+n+1}, \quad [M_m, Y_{n+\frac{1}{2}}] = 0.$$

This infinite-dimensional Lie algebra was originally introduced in [3] by looking at the invariance of the free Schrödinger equation in $(1+1)$-dimensions $(2i\mathcal{M}\partial_t - \partial_r^2)\psi = 0$. It is easy to see that $\mathfrak{sv}$ is a semi-direct product of Witt algebra $\mathfrak{Vir}_o = \bigoplus \mathbb{C}L_n$ and the two-step nilpotent infinite-dimensional Lie algebra $\mathfrak{h} = \bigoplus \mathbb{C}M_n \bigoplus \mathbb{C}Y_{n+\frac{1}{2}}$. The structure and representation theory of $\mathfrak{sv}$ have been studied

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by C. Roger and J. Unterberger in [12]. The irreducible weight modules with finite-dimensional weight spaces over $\mathfrak{sv}$ are classified in [7].

In order to investigate the vertex representations of $\mathfrak{sv}$, J. Unterberger (see [15]) introduced a class of infinite-dimensional Lie algebras $\widetilde{\mathfrak{sv}}$ called the extended Schrödinger–Virasoro Lie algebra, which can be viewed as an extension of $\mathfrak{sv}$ by a conformal current with conformal weight 1. The extended Schrödinger–Virasoro Lie algebra $\widetilde{\mathfrak{sv}}$ is a vector space spanned by a basis $\{L_n, M_n, N_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}$, with commutation relations

$$[N_m, N_n] = 0, \qquad [N_m, M_n] = 2M_{m+n},$$
$$[L_m, N_n] = nN_{n+m}, \qquad [N_m, Y_{n+\frac{1}{2}}] = Y_{m+n+\frac{1}{2}},$$

and the other relations are the same as those of $\mathfrak{sv}$. For all $m, n \in \mathbb{Z}$. The structure of $\widetilde{\mathfrak{sv}}$ has been studied in [2].

Recently, a number of new classes of infinite-dimensional Lie algebras over a field of characteristic 0 related to the Virasoro algebra were studied by several authors (see [6,8,9,11,13,14,16]). Among those algebras, are the generalized Witt algebras, the generalized Virasoro algebras, the Lie algebras of generalized Weyl type and the generalized Heisenberg–Virasoro algebra. In the present paper we define and study a new class of infinite-dimensional Lie algebras which contains the Schrödinger–Virasoro Lie algebra $\mathfrak{sv}$ as a special case.

Let $\mathbb{F}$ be a field of characteristic 0, G an additive subgroup of $\mathbb{F}$, $\alpha \in \mathbb{F}$ such that $\alpha \notin G$, $2\alpha \in G$. Motivated by the above algebras, we introduce a new class of Lie algebras which we call the generalized Schrödinger–Virasoro Lie algebras. We use $\mathfrak{gsv}[G,\alpha]$ to denote the one corresponding to such a group G and element α. In this paper we study the automorphism group $\mathrm{Aut}(\mathfrak{gsv}[G,\alpha])$ and the irreducibility of Verma modules over the generalized Schrödinger–Virasoro Lie algebra $\mathfrak{gsv}[G,\alpha]$.

The paper is organized as follows. In Section 2, we introduce the generalized Schrödinger–Virasoro Lie algebra $\mathfrak{gsv}[G,\alpha]$. The necessary and sufficient conditions of isomorphism between two of these algebras are determined. In Section 3, we determine the automorphism group of $\mathfrak{gsv}[G,\alpha]$. In Section 4, a Verma module $V(c,h)$ over the generalized Schrödinger–Virasoro Lie algebra $\mathfrak{gsv}[G,\alpha]$ is defined and its irreducibility is completely determined.

Throughout the article, we denote by $\mathbb{Z}$ the set of integers, $\mathbb{N}$ the set of non-negative integers.

2. Generalized Schrödinger–Virasoro algebras

Let $\mathbb{F}$ be a field of characteristic 0, G an additive proper subgroup of $\mathbb{F}$, $\alpha \in \mathbb{F}$ satisfying $\alpha \notin G$ while $2\alpha \in G$. We set $G_1 = \alpha + G$ and $T = G \cup G_1$. It is obvious that T is an additive subgroup of $\mathbb{F}$. In this section we want to make a natural generalization of Schrödinger–Virasoro algebra $\mathfrak{sv}$, this leads us to the following definition.

Definition 2.1. The generalized Schrödinger–Virasoro algebra $\mathfrak{gsv}[G,\alpha]$ is defined to be the Lie algebra with $\mathbb{F}$ basis $\{L_u, M_u, Y_{u+\alpha} \mid u \in G\}$ subject to the following Lie brackets:

$$[L_u, L_v] = (v-u)L_{u+v}, \quad [L_u, M_v] = vM_{u+v}, \quad [L_u, Y_{v+\alpha}] = \left(v+\alpha-\frac{u}{2}\right)Y_{u+v+\alpha},$$
$$[M_u, M_v] = 0, \quad [Y_{u+\alpha}, Y_{v+\alpha}] = (v-u)M_{u+v+2\alpha}, \quad [M_u, Y_{v+\alpha}] = 0.$$

It is straightforward to see that $\mathfrak{gsv}[G,\alpha]$ is T-graded:

$$\mathfrak{gsv}[G,\alpha] = \bigoplus_{x \in T} \mathfrak{gsv}[G,\alpha]_x,$$

where

$$\mathfrak{gsv}[G,\alpha]_x = \begin{cases} \mathbb{F}L_x \oplus \mathbb{F}M_x, & x \in G, \\ \mathbb{F}Y_x, & x \in G_1. \end{cases}$$

The homogeneous spaces are the root spaces according to the Cartan subalgebra $\mathfrak{gsv}[G,\alpha]_0$.

One can see that if $G = \mathbb{Z}$ and $\alpha = \frac{1}{2}$, then the generalized Schrödinger–Virasoro algebra $\mathfrak{gsv}[G,\alpha]$ is nothing but the Schrödinger–Virasoro algebra $\mathfrak{sv}$ defined by Henkel in [3].

Denote $L = \bigoplus_{u\in G} \mathbb{F}L_u$, $M = \bigoplus_{u\in G} \mathbb{F}M_u$, $Y = \bigoplus_{v\in G} \mathbb{F}Y_{\alpha+v}$, $I = M \oplus Y$. Obviously, L is the generalized Witt algebra (see [13]). M and I are ideals of L.

Lemma 2.2. *I is the unique maximal ideal of $\mathfrak{gsv}[G,\alpha]$.*

Proof. It is obvious that I is an ideal of $\mathfrak{gsv}[G,\alpha]$. Moreover, I is a maximal ideal of $\mathfrak{gsv}[G,\alpha]$ since $\mathfrak{gsv}[G,\alpha]/I$ is a simple generalized Witt algebra.

Now suppose I_1 is another maximal ideal of $\mathfrak{gsv}[G,\alpha]$, we need to prove $I_1 = I$, $\forall z \in I_1$, $z \neq 0$, suppose $z = l + x + y$, where $l \in L$, $x \in M$, $y \in Y$. Assume $l \neq 0$. It is obvious that $0 \neq ad\, M_v(z) \in M \cap I_1$ for any $0 \neq v \in G$ by using the Lie bracket of $\mathfrak{gsv}[G,\alpha]$. Suppose

$$z_0 = ad\, M_v(z) = a_1 M_{u_1} + a_2 M_{u_2} + \cdots + a_n M_{u_n}.$$

It is well known that the submodules of a weight module are also weight modules (see [5]), thus all homogeneous components of z_0 are contained in I_1, hence we can find some element $0 \neq u \in G$ such that $M_u \in I_1$. Thus $M \subseteq I_1$ due to the fact $[L_v, M_u] = uM_{u+v} \in I_1$ for any $v \in G$. Then

$$0 \neq z' = l + y \in I_1.$$

If $y = 0$, $z' = l \in I_1$, this implies that $I_1 = \mathfrak{gsv}[G,\alpha]$, which is a contradiction. Suppose $y \neq 0$. Since $[Y_{\alpha+v}, l + y] \subseteq (M \oplus Y) \cap I_1$ for any $v \in G$ and $M \subseteq I_1$, we claim that there exists some $0 \neq y' \in I_1 \cap Y$. Then all homogeneous components of y' are contained in I_1. This implies that $Y \subset I_1, l \in I_1$, and therefore $L \subset I_1$. Therefore $I_1 = \mathfrak{gsv}[G,\alpha]$, which is a contradiction. Thus $l = 0$, $z = m + y$. So $I_1 \subseteq M \oplus Y = I$. By the maximality of I and I_1, we have $I_1 = I$. $\square$

Let G' be another additive proper subgroup of $\mathbb{F}$, $\alpha' \in \mathbb{F}$, such that $\alpha' \notin G'$, $2\alpha' \in G'$. Denote $G'_1 = \alpha' + G'$, $T' = G' \cup G'_1$.

Corresponding to G and G', there are two generalized Virasoro algebras: $Vir[G]$ and $Vir[G']$. About $Vir[G]$ and $Vir[G']$, it was pointed out in [13] that the following lemma can be obtained by using Theorem 4.2 in [1]. However, it can be proved straightforwardly.

Lemma 2.3. *(See [13].) $Vir[G] \simeq Vir[G']$ if and only if there exists $a \in \mathbb{F}^*$ such that $aG = G'$.*

Proof. Since $Vir[G] \simeq Vir[G'] \Leftrightarrow Vir[G]/C \simeq Vir[G']/C'$, where C and C' are the center of $Vir[G]$ and $Vir[G']$ respectively, we view $Vir[G]$ and $Vir[G']$ as generalized Witt algebras. Let $\theta : Vir[G] \to Vir[G']$ be an isomorphism of Lie algebras. Since $\mathbb{F}L_0$ and $\mathbb{F}L'_0$ are the unique Cartan subalgebras of $Vir[G]$ and $Vir[G']$ respectively, there exists $a \in \mathbb{F}^*$ such that $\theta(L_0) = aL'_0$. By applying θ to the relation $[L_0, L_x] = xL_x$, we have $x\theta(L_x) = [\theta(L_0), \theta(L_x)] = a[L'_0, \theta(L_x)]$, so

$$\left[L'_0, \theta(L_x)\right] = a^{-1} x \theta(L_x).$$

Thus $a^{-1}x \in G'$ since G' is the weight set according to the unique Cartan algebra $\mathbb{F}L'_0$. Hence $a^{-1}G \subseteq G'$. Similarly, by applying θ^{-1} to the bracket $[L'_0, L'_{x'}]$, we get $aG' \subseteq G$. So, $aG' = G$.

The sufficiency is obvious. This completes the proof of Lemma 2.3. $\square$

Theorem 2.4. *$\mathfrak{gsv}[G,\alpha]$ and $\mathfrak{gsv}[G',\alpha']$ are isomorphic if and only if there exists $a \in \mathbb{F}^*$ such that $G' = aG$, $T' = aT$.*

Proof. Let $\theta : \mathfrak{gsv}[G,\alpha] \to \mathfrak{gsv}[G',\alpha']$ be an isomorphism of Lie algebras, I, I' be the maximal ideals of $\mathfrak{gsv}[G,\alpha]$ and $\mathfrak{gsv}[G',\alpha']$ respectively. By Lemma 2.2, we have $\theta(I) = I'$, thus there is an isomorphism of the generalized Witt algebras induced by θ:

$$\bar{\theta} : \mathfrak{gsv}[G,\alpha]/I \to \mathfrak{gsv}[G',\alpha']/I'.$$

By applying Lemma 2.3, there exists $a \in \mathbb{F}^*$ such that

$$G' = aG, \tag{1}$$

and $\bar{\theta}(\bar{L}_u) = \chi(u)a^{-1}\bar{L}'_{au}$, for some $\chi \in \mathrm{Hom}(G,\mathbb{F}^*)$. So we can assume

$$\theta(L_u) = \chi(u)a^{-1}L'_{au} + m'^L_u + y'^L_u. \tag{2}$$

Suppose $\theta(Y_{\alpha+v}) = \sum_{i=1}^{s} c_{v'_i} Y'_{\alpha'+v'_i} + m'^Y_v$. By using θ to both sides of the identity $(\alpha+v)Y_{\alpha+v} = [L_0, Y_{\alpha+v}]$, we have

$$\begin{aligned}
(\alpha+v)\theta(Y_{\alpha+v}) &= \big[\theta(L_0), \theta(Y_{\alpha+v})\big] \\
&= \left[a^{-1}L'_0 + m'^L_0 + y'^L_0, \sum_{i=1}^{s} c_{v'_i} Y'_{\alpha'+v'_i} + m'^Y_v\right] \\
&= \left[a^{-1}L'_0 + y'^L_0, \sum_{i=1}^{s} c_{v'_i} Y'_{\alpha'+v'_i} + m'^Y_v\right] \\
&= a^{-1}\sum_{i=1}^{s} c_{v'_i}\big[L'_0, Y'_{\alpha'+v'_i}\big] + m' \\
&= a^{-1}\sum_{i=1}^{s} c_{v'_i}\big(\alpha' + v'_i\big)Y'_{\alpha'+v'_i} + m'
\end{aligned}$$

for some $m' \in M'$. Then we have

$$a^{-1}\sum_{i=1}^{s} c_{v'_i}\big(\alpha' + v'_i\big)Y'_{\alpha'+v'_i} + m' = (\alpha+v)\left(\sum_{i=1}^{s} c_{v'_i} Y'_{\alpha'+v'_i} + m'^Y_v\right).$$

By comparing the coefficients, we get

$$v'_i = -\alpha' + a(\alpha+v), \quad \forall i \in \{1,\ldots,s\}.$$

Hence $s = 1$ and

$$\theta(Y_{\alpha+v}) = c'_v Y'_{a(\alpha+v)} + m'^Y_v, \tag{3}$$

where $c'_v = c_{-\alpha'+a(\alpha+v)}$.

As we know that $\theta(I)=I'$, M and M' are the centers of I and I' respectively, there exists an induced isomorphism

$$\bar{\bar{\theta}}:\bar{Y}=I/M\to I'/M'=\bar{Y}':\qquad \bar{Y}_{\alpha+v}\mapsto c'_v\bar{Y}'_{a(\alpha+v)}.$$

Thus we have the following isomorphisms of vector spaces:

$$Y\xrightarrow{\pi}\bar{Y}\xrightarrow{\bar{\bar{\theta}}}\bar{Y'}\xrightarrow{\pi^{-1}}Y'$$

where π and π' are the canonical homomorphisms of vector spaces. Thus

$$\pi'^{-1}\bar{\bar{\theta}}\pi:\bigoplus_{v\in G}Y_{\alpha+v}=Y\to Y'=\bigoplus_{v'\in G'}Y_{\alpha'+v'}:\qquad Y_{\alpha+v}\mapsto c'_vY'_{a(\alpha+v)}$$

is an isomorphism of vector spaces. Thus $\alpha'+G'=a(\alpha+G)$. This and (1) give us

$$G'=aG,\qquad T'=aT.$$

On the other hand, if $G'=aG$, $T'=aT$, we define a map

$$\theta:\mathfrak{gsv}[G,\alpha]\to\mathfrak{gsv}[G',\alpha']:\quad L_u\mapsto a^{-1}L'_{au},\ M_u\mapsto a^{-1}M'_{au},\ Y_{\alpha+u}\mapsto a^{-1}Y'_{a(\alpha+u)}.$$

It is straightforward to check that θ is an isomorphism of Lie algebras. This completes the proof of Theorem 2.4. □

Corollary 2.5. *The map*:

$$\begin{aligned}\theta:\mathfrak{sv}=\mathfrak{gsv}\left[\mathbb{Z},\frac{1}{2}\right]&\to\mathfrak{gsv}[G,\alpha]:\\ L_i&\mapsto(2\alpha)^{-1}L_{2\alpha i},\\ M_i&\mapsto(2\alpha)^{-1}M_{2\alpha i},\\ Y_{i+\frac{1}{2}}&\mapsto(2\alpha)^{-1}Y_{2\alpha i+\alpha}\end{aligned}$$

extends uniquely to a Lie algebra isomorphism between $\mathfrak{sv}=\mathfrak{gsv}[\mathbb{Z},\frac{1}{2}]$ *and* $\mathfrak{gsv}[2\mathbb{Z}\alpha,\alpha]$.

Lemma 2.6. *Let* G,α,G',α' *be as in Theorem* 2.4, $\theta:\mathfrak{gsv}[G,\alpha]\to\mathfrak{gsv}[G',\alpha']$ *be a Lie algebra isomorphism. Then*

$$\begin{cases}\theta(L_u)=\chi(u)a^{-1}L'_{au}+m'^L_u+y'^L_u,\\ \theta(M_u)=b_uM'_{au},\\ \theta(Y_{\alpha+v})=c_vY'_{a(\alpha+v)}+m'^Y_v\end{cases}$$

for some $b_u,c_v\in\mathbb{F}^*$, $m'^Y_v,m'^L_u\in M'$, $y'^L_u\in Y'$ *and* $\chi\in\mathrm{Hom}(G,\mathbb{F}^*)$, *where* $a\in\mathbb{F}^*$ *such that* $aG=G'$, $aT=T'$.

Proof. The first and the third identities are given in the proof of Theorem 2.4 (i.e., identities (2) and (3)), we only need to prove the second formula. Since $\mathbb{F}M_0$ (resp. $\mathbb{F}M_0'$) is the center of $\mathfrak{gsv}[G,\alpha]$ (resp. $\mathfrak{gsv}[G',\alpha']$), we have

$$\theta(M_0)=b_0M_0'$$

for some $b_0\in\mathbb{F}^*$. Recall that $\theta(I)=I'$, $\theta(M)=M'$, for $u\neq 0$, suppose

$$\theta(M_u)=\sum_{i=1}^{n}b_{u_i'}M_{u_i'}'+b_0'M_0',$$

where $u_i\neq 0$, $b_{u_i}\neq 0$. Then we have

$$\begin{aligned}
u\theta(M_u)&=\theta\big([L_0,M_u]\big)\\
&=\left[a^{-1}L_0'+m_0'^{L}+y_0'^{L},\sum_{i=1}^{n}b_{u_i'}M_{u_i'}'+b_0'M_0'\right]\\
&=\left[a^{-1}L_0',\sum_{i=1}^{n}b_{u_i'}M_{u_i'}'\right]=a^{-1}\sum_{i=1}^{n}b_{u_i'}\big[L_0',M_{u_i'}'\big]\\
&=a^{-1}\sum_{i=1}^{n}b_{u_i'}u_i'M_{u_i'}'.
\end{aligned}$$

So we have

$$u\left(\sum_{i=1}^{n}b_{u_i'}M_{u_i'}'+b_0'M_0'\right)=a^{-1}\sum_{i=1}^{n}b_{u_i'}u_i'M_{u_i'}'.$$

Thus $b_0'=0$, $\sum_{i=1}^{n}(ub_{u_i'}-a^{-1}b_{u_i'}u_i')M_{u_i}'=0$. Hence $u_i'=au$, $\forall i\in\{1,2,\ldots,n\}$, and $n=1$. This gives

$$\theta(M_u)=b_uM_{au}',\quad \forall u\in G.\qquad\square$$

3. Automorphism group of $\mathfrak{gsv}[G,\alpha]$

In this section, we first construct three kinds of outer automorphisms of $\mathfrak{gsv}[G,\alpha]$, then determine the automorphism group of $\mathfrak{gsv}[G,\alpha]$ completely. Throughout this section we always assume that $\mathbb{F}$ is an algebraic closed field with characteristic 0. We remark that the space of outer derivations of $\mathfrak{sv}$ is determined in [12, Theorem 4.6].

Lemma 3.1.

(i) *For any $\chi\in\mathrm{Hom}(T,\mathbb{F}^*)$ and $b\in\mathbb{F}^*$, the map*

$$\begin{gathered}
\sigma_b^{\chi}:\mathfrak{gsv}[G,\alpha]\to\mathfrak{gsv}[G,\alpha]:\\
L_u\mapsto\chi(u)L_u,\\
M_u\mapsto b\chi(u)M_u,\\
Y_{\alpha+u}\mapsto b^{\frac{1}{2}}\chi(\alpha+u)Y_{\alpha+u}
\end{gathered}$$

is an automorphism of $\mathfrak{gsv}[G,\alpha]$. *Furthermore, the set* $\{\sigma_b^\chi \mid \chi \in \mathrm{Hom}(T,\mathbb{F}^*),\ b \in \mathbb{F}^*\}$ *forms a subgroup of* $\mathrm{Aut}(\mathfrak{gsv}[G,\alpha])$ *and this subgroup is isomorphic to the direct product* $(\mathrm{Hom}(T,\mathbb{F}^*) \times \mathbb{F}^*)$, *namely,* $\sigma_{b_1}^{\chi_1}\sigma_{b_2}^{\chi_2} = \sigma_{b_1b_2}^{\chi_1\chi_2}$ *for* $b_1, b_2 \in \mathbb{F}^*$ *and* $\chi_1, \chi_2 \in \mathrm{Hom}(T,\mathbb{F}^*)$.

(ii) *For any* $a \in S(G,T) := \{a \in \mathbb{F}^* \mid aG = G,\ aT = T\}$, *the map*

$$\begin{aligned}\varphi_a : \mathfrak{gsv}[G,\alpha] &\to \mathfrak{gsv}[G,\alpha]:\\ L_u &\mapsto a^{-1}L_{au},\\ M_u &\mapsto a^{-1}M_{au},\\ Y_{\alpha+u} &\mapsto a^{-1}Y_{a(\alpha+u)}\end{aligned}$$

is an automorphism of $\mathfrak{gsv}[G,\alpha]$, *and* $\{\varphi_a \mid a \in S(G,T)\}$ *forms a subgroup of* $\mathrm{Aut}(\mathfrak{gsv}[G,\alpha])$, *where* $\varphi_a\varphi_b = \varphi_{ab}$ *for* $a, b \in \mathbb{F}^*$.

(iii) *Denote*

$$\mathcal{A} := \bigl\{\underline{a} = (a_u)_{u\in G} \bigm| (v-u)a_{u+v} = va_v - ua_u,\ \forall u, v \in G\bigr\}.$$

The map $\phi_{\underline{a}} : \mathfrak{gsv}[G,\alpha] \to \mathfrak{gsv}[G,\alpha]$:

$$L_u \mapsto L_u + a_uM_u, \quad M_u \mapsto M_u, \quad Y_{\alpha+v} \mapsto Y_{\alpha+v}$$

is an automorphism. $\Phi = \{\phi_{\underline{a}} \mid \underline{a} \in \mathcal{A}\}$ *forms a subgroup of* $\mathrm{Aut}(\mathfrak{gsv}[G,\alpha])$, *where* $\phi_{\underline{a}}\phi_{\underline{b}} = \phi_{\underline{a+b}}$ *for* $\underline{a}, \underline{b} \in \mathcal{A}$.

Proof. The proof is straightforward, we omit the details. □

Theorem 3.2. *Let* $\mathrm{Inn}(\mathfrak{gsv}[G,\alpha])$ *be the inner automorphism subgroup of* $\mathrm{Aut}(\mathfrak{gsv}[G,\alpha])$. *Then we have*

$$\mathrm{Aut}\bigl(\mathfrak{gsv}[G,\alpha]\bigr) \simeq \Bigl(\Bigl(\bigl(\mathrm{Hom}\bigl(T,\mathbb{F}^*\bigr) \times \mathbb{F}^*\bigr) \rtimes S(G,T)\Bigr) \ltimes \Phi\Bigr) \ltimes \mathrm{Inn}\bigl(\mathfrak{gsv}[G,\alpha]\bigr).$$

Proof. For any $\theta \in \mathrm{Aut}(\mathfrak{gsv}[G,\alpha])$, by Lemma 2.6, we can assume

$$\begin{aligned}\theta(L_u) &= \chi_1(u)a^{-1}L_{au} + m_u^L + y_u^L,\\ \theta(M_u) &= \chi'(u)a^{-1}M_{au},\\ \theta(Y_{\alpha+v}) &= \chi'(\alpha+v)a^{-1}Y_{a(\alpha+v)} + m_v^Y,\end{aligned}$$

where $\chi_1 \in \mathrm{Hom}(G,\mathbb{F}^*)$, $\chi' : T \to \mathbb{F}^*$ is a map, $a \in S(G,T)$, $m_u^L, m_v^Y \in M$, $y_u^L \in Y$.

By applying θ to both sides of the identities:

$$\begin{aligned}[L_{-u}, M_u] &= uM_0,\\ [L_u, Y_{\alpha+v}] &= \left(\alpha + v - \frac{u}{2}\right)Y_{\alpha+v+u},\end{aligned}$$

and

$$[Y_{\alpha+u}, Y_{\alpha+v}] = (v-u)M_{u+v+2\alpha},$$

we obtain

$$\chi'(u) = b\chi_1(u), \quad \forall u \in G, \tag{4}$$

where $b = \chi'(0) \in \mathbb{F}^*$,

$$\chi_1(u)\chi'(\alpha + v) = \chi'(u + v + \alpha), \tag{5}$$

and

$$\chi'(\alpha + u)\chi'(\alpha + v) = \chi'(u + v + 2\alpha) \tag{6}$$

for $u, v \in G$.

Write

$$\chi(x) = \begin{cases} \chi_1(x), & \text{if } x \in G, \\ b^{-\frac{1}{2}}\chi'(x), & \text{if } x \in G_1 = \alpha + G, \end{cases}$$

and by using (4)–(6), one can easily see that $\chi \in \mathrm{Hom}(T, \mathbb{F}^*)$. By using Lemma 3.1(i),

$$\begin{aligned} \sigma_b^\chi : \mathfrak{gsv}[G,\alpha] &\to \mathfrak{gsv}[G,\alpha]: \\ L_u &\mapsto \chi(u)L_u, \\ M_u &\mapsto b\chi(u)M_u, \\ Y_{\alpha+v} &\mapsto b^{\frac{1}{2}}\chi(\alpha+v)Y_{\alpha+v} \end{aligned}$$

is an automorphism of $\mathfrak{gsv}[G,\alpha]$.

Since $(\sigma_b^\chi)^{-1}\theta \in \mathrm{Aut}(\mathfrak{gsv}[G,\alpha])$, it is obvious that

$$\begin{aligned} \left(\sigma_b^\chi\right)^{-1}\theta : \mathfrak{gsv}[G,\alpha] &\to \mathfrak{gsv}[G,\alpha]: \\ L_u &\mapsto a^{-1}L_{au} + m_{1u}^L + y_{1u}^L, \\ M_u &\mapsto a^{-1}M_{au}, \\ Y_{\alpha+v} &\mapsto a^{-1}Y_{a(\alpha+v)} + m_{1v}^Y, \end{aligned}$$

where $m_{1u}^L, m_{1v}^Y \in M$, $y_{1u}^L \in Y$.

Set

$$\begin{aligned} \varphi_a : \mathfrak{gsv}[G,\alpha] &\to \mathfrak{gsv}[G,\alpha]: \\ L_u &\mapsto a^{-1}L_{au}, \\ M_u &\mapsto a^{-1}M_{au}, \\ Y_{\alpha+v} &\mapsto a^{-1}Y_{a(\alpha+v)}. \end{aligned}$$

By Lemma 3.1(ii), $\varphi_a \in \mathrm{Aut}(\mathfrak{gsv}[G,\alpha])$. Set $\tau = (\varphi_a)^{-1}(\sigma_b^\chi)^{-1}\theta$, then $\tau \in \mathrm{Aut}(\mathfrak{gsv}[G,\alpha])$. More precisely

$$\tau(L_u) = L_u + m^L_{2u} + y^L_{2u}, \quad \tau(M_u) = M_u, \quad \tau(Y_{\alpha+v}) = Y_{\alpha+v} + m^Y_{2v}, \tag{7}$$

where $m^L_{2u}, m^Y_{2v} \in M$, $y^L_{2u} \in Y$.

Claim. $\tau = (\varphi_a)^{-1}(\sigma^{\chi}_{b})^{-1}\theta \in \mathrm{Inn}(\mathfrak{gsv}[G,\alpha]) \cdot \Phi$.

In fact, assume

$$\tau(L_0) = L_0 + \sum_{i=1}^{p} a_{u_i} M_{u_i} + \sum_{j=1}^{q} b_{v_j} Y_{\alpha+v_j} + b_0 Y_\alpha, \qquad \tau(Y_\alpha) = Y_\alpha + \sum_{k=1}^{r} c_{w_k} M_{w_k}.$$

Applying τ to $[L_0, Y_\alpha] = \alpha Y_\alpha$, we have

$$\sum_{k=1}^{r} c_{w_k} w_k M_{w_k} - \sum_{j=1}^{q} b_{v_j} v_j M_{2\alpha+v_j} = \alpha \sum_{k=1}^{r} c_{w_k} M_{w_k}.$$

Noting that $v_j \neq 0$, $\alpha \neq w_k$, we have

$$q = r, \quad v_k = w_k - 2\alpha, \quad b_{v_k} = \frac{c_{w_k(w_k-\alpha)}}{w_k - 2\alpha}.$$

These gives us

$$\tau(L_0) = L_0 + \sum_{i=1}^{p} a_{u_i} M_{u_i} + \sum_{k=1}^{r} \frac{c_{w_k}(w_k - \alpha)}{w_k - 2\alpha} Y_{w_k-\alpha} + b_0 Y_\alpha. \tag{8}$$

Now we construct an inner automorphism τ' for $\mathfrak{gsv}[G,\alpha]$, which is equal to τ when acting on L_0 and Y_α. Indeed, we set

$$\begin{aligned}\tau' = \exp ad\Bigg(& \sum_{1 \leqslant j \neq k \leqslant r} \frac{c_{w_k} c_{w_j} (\alpha - w_j)(w_j - w_k)}{2(w_k - 2\alpha)(w_j - 2\alpha)(w_j + w_k - 2\alpha)} M_{w_j+w_k-2\alpha} \\ & + \sum_{k=1}^{r} \frac{b_0 c_{w_k}(2\alpha - w_k)}{2\alpha w_k} M_{w_k}\Bigg) \\ & \times \exp ad\left(-\sum_{i=1}^{p} \frac{a_{u_i}}{u_i} M_{u_i}\right) \exp ad\left(\sum_{k=1}^{r} \frac{c_{w_k}}{2\alpha - w_k} Y_{w_k-\alpha} - \frac{b_0}{\alpha} Y_\alpha\right).\end{aligned}$$

Then one can check that the inner automorphism τ' satisfies

$$\tau'(L_0) = \tau(L_0), \qquad \tau'(Y_\alpha) = \tau(Y_\alpha).$$

For any $u \in G$, we apply τ' to $[L_0, L_u] = uL_u$. For the right side, we have

$$u\tau'(L_u) = u\tau(L_u) + u\big(\tau'(L_u) - \tau(L_u)\big).$$

For the left side, we have

$$\begin{aligned}\big[\tau'(L_0), \tau'(L_u)\big] &= \big[\tau(L_0), \tau'(L_u)\big] = \big[\tau(L_0), \tau(L_u) + \big(\tau'(L_u) - \tau(L_u)\big)\big] \\ &= u\tau(L_u) + \big[\tau(L_0), \tau'(L_u) - \tau(L_u)\big].\end{aligned}$$

Thus

$$\big[\tau(L_0), \tau'(L_u) - \tau(L_u)\big] = u\big(\tau'(L_u) - \tau(L_u)\big). \tag{9}$$

Now we prove the following identity.

$$\tau'(L_u) = \tau(L_u) + e_u M_u \tag{10}$$

for some $e_u \in \mathbb{F}$.

Indeed, by using (7) and the definition of τ', one can see that

$$\tau'(M_u) = \tau(M_u) = M_u. \tag{11}$$

So $[\tau'(L_u), M_v] = vM_{u+v}$. Thus $\tau'(L_u) = L_u + m_1 + y_1$ for some $m_1 \in M$, $y_1 \in Y$. By using (7) again, we have $\tau'(L_u) - \tau(L_u) \in M \oplus Y$. Write

$$\tau'(L_u) - \tau(L_u) = \sum_{k=1}^{r} e_{v_k} M_{v_k} + \sum_{l=1}^{s} d_{w_l} Y_{\alpha+w_l}.$$

By using this along with identities (8) and (9), we have

$$\begin{aligned}&\left[L_0 + \sum_{i=1}^{p} a_{u_i} M_{u_i} + \sum_{j=1}^{q} b_{v_j} Y_{\alpha+v_j} + b_0 Y_\alpha, \sum_{k=1}^{r} e_{v_k} M_{v_k} + \sum_{l=1}^{s} d_{w_l} Y_{\alpha+w_l}\right] \\ &\quad= \sum_{k=1}^{r} e_{v_k} v_k M_{v_k} + \sum_{l=1}^{s} d_{w_l}(\alpha + w_l) Y_{\alpha+w_l} \\ &\qquad+ \sum_{j=1}^{q} \sum_{l=1}^{s} b_{v_j} d_{w_l}(w_l - v_j) M_{2\alpha+w_l+v_j} + \sum_{l=1}^{s} b_0 d_{w_l} w_l M_{2\alpha+w_l} \\ &\quad= u\left(\sum_{k=1}^{r} e_{v_k} M_{v_k} + \sum_{l=1}^{s} d_{w_l} Y_{\alpha+w_l}\right).\end{aligned}$$

By comparing the coefficients of $Y_{\alpha+w_l}$ we have $ud_{w_l} = d_{w_l}(\alpha + w_l)$. This means $d_{w_l} = 0$ for any $l \in \{1, \dots, s\}$ since $u \neq \alpha + w_l$. Thus the we get

$$\sum_{k=1}^{r} e_{v_k} v_k M_{v_k} = u\left(\sum_{k=1}^{r} e_{v_k} M_{v_k}\right).$$

Hence, $r = 1$, $u = v_1$. Thus $\tau'(L_u) - \tau(L_u) = e_u M_u$. This proves (10).

For any $v \in G$, $v \neq 2\alpha$, by (10), we have

$$\tau'(Y_{\alpha+v}) = \left(\alpha - \frac{v}{2}\right)^{-1} \tau'[L_v, Y_\alpha] = \left(\alpha - \frac{v}{2}\right)^{-1} \big[\tau(L_v) + e_v M_v, \tau(Y_\alpha)\big] = \tau(Y_{\alpha+v}).$$

For $v = 2\alpha$, we have

$$\tau'(Y_{3\alpha}) = (-3\alpha)^{-1}\big[\tau'(L_{4\alpha}), \tau'(Y_{-\alpha})\big] = (-3\alpha)^{-1}\big[\tau(L_{4\alpha}) + b_{4\alpha} M_{4\alpha}, \tau(Y_{-\alpha})\big] = \tau(Y_{3\alpha}).$$

In all cases we get

$$\tau'(Y_{\alpha+v}) = \tau(Y_{\alpha+v}), \quad \forall v \in G. \tag{12}$$

Define

$$\phi : \mathfrak{gsv}[G,\alpha] \to \mathfrak{gsv}[G,\alpha]: \quad L_u \mapsto L_u + e_u M_u,\ M_u \mapsto M_u,\ Y_{\alpha+v} \mapsto Y_{\alpha+v}.$$

It is easy to check that ϕ is an automorphism in the group Φ. By (10)–(12), we have $\tau' = \tau\phi$. Thus $\tau = \tau'\phi^{-1} \in \mathrm{Inn}(\mathfrak{gsv}[G,\alpha]) \cdot \Phi$, which completes the proof of the claim.

By the claim, we have

$$\theta = \sigma_b^\chi \varphi_a \tau' \phi^{-1} \in \big(\mathrm{Hom}\big(T, \mathbb{F}^*\big) \times \mathbb{F}^*\big) \cdot S(G,T) \cdot \mathrm{Inn}\big(\mathfrak{gsv}[G,\alpha]\big) \cdot \Phi.$$

Since $\mathrm{Inn}(\mathfrak{gsv}[G,\alpha])$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{gsv}[G,\alpha])$, thus $\mathrm{Inn}(\mathfrak{gsv}[G,\alpha]) \cdot \Phi = \Phi \cdot \mathrm{Inn}(\mathfrak{gsv}[G,\alpha])$. One can check straightforwardly that the following two facts hold:

$$\big(\mathrm{Hom}\big(T, \mathbb{F}^*\big) \times \mathbb{F}^*\big) \lhd \big(\mathrm{Hom}\big(T, \mathbb{F}^*\big) \times \mathbb{F}^*\big) \cdot S(G,T),$$

and

$$\Phi \lhd \big(\mathrm{Hom}\big(T, \mathbb{F}^*\big) \times \mathbb{F}^*\big) \cdot S(G,T) \cdot \Phi.$$

Thus

$$\mathrm{Aut}\big(\mathfrak{gsv}[G,\alpha]\big) \simeq \Big(\Big(\big(\mathrm{Hom}\big(T, \mathbb{F}^*\big) \times \mathbb{F}^*\big) \rtimes S(G,T)\Big) \ltimes \Phi\Big) \ltimes \mathrm{Inn}\big(\mathfrak{gsv}[G,\alpha]\big).$$

This completes the proof of Theorem 3.2. $\square$

4. Verma modules of $\mathfrak{gsv}[G,\alpha]$

In this section we construct and investigate the structure of Verma modules over the generalized Schrödinger–Virasoro algebra $\mathfrak{gsv}[G,\alpha]$.

Note that $T = G \cup G_1$ is a subgroup of $\mathbb{F}$, we fix a total order "$\succcurlyeq$" on T which is compatible with the addition, i.e., $x \succcurlyeq y$ implies $x + z \succcurlyeq y + z$ for any $z \in T$ (see [4,10]). We write $x \succ y$ if $x \succcurlyeq y$ and $x \neq y$. Let

$$T_+ := \{x \in T \mid x \succ 0\}, \qquad T_- := \{x \in T \mid x \prec 0\}.$$

Then $T = T_+ \cup \{0\} \cup T_-$ and $\mathfrak{gsv}[G,\alpha]$ has a triangular decomposition:

$$\mathfrak{gsv}[G,\alpha] = \mathfrak{gsv}[G,\alpha]_- \oplus \mathfrak{gsv}[G,\alpha]_0 \oplus \mathfrak{gsv}[G,\alpha]_+,$$

where

$$\mathfrak{gsv}[G,\alpha]_- = \bigoplus_{u\prec 0}\mathbb{F}L_u \oplus \bigoplus_{u\prec 0}\mathbb{F}M_u \oplus \bigoplus_{\alpha+v\prec 0}\mathbb{F}Y_{\alpha+v},$$

$$\mathfrak{gsv}[G,\alpha]_+ = \bigoplus_{u\succ 0}\mathbb{F}L_u \oplus \bigoplus_{u\succ 0}\mathbb{F}M_u \oplus \bigoplus_{\alpha+v\succ 0}\mathbb{F}Y_{\alpha+v}$$

and $\mathfrak{gsv}[G,\alpha]_0 = \mathbb{F}L_0 \oplus \mathbb{F}M_0$. The universal enveloping algebra of $\mathfrak{gsv}[G,\alpha]$ is given by

$$U\big(\mathfrak{gsv}[G,\alpha]\big) = U\big(\mathfrak{gsv}[G,\alpha]\big)_- U\big(\mathfrak{gsv}[G,\alpha]\big)_0 U\big(\mathfrak{gsv}[G,\alpha]\big)_+.$$

The elements $L_{i_1}\cdots L_{i_r}M_{j_1}\cdots M_{j_s}Y_{\alpha+k_1}\cdots Y_{\alpha+k_t}$, where $r,s,t\in\mathbb{N}$, $i_1\succcurlyeq\cdots\succcurlyeq i_r$, $j_1\succcurlyeq\cdots\succcurlyeq j_s$, $k_1\succcurlyeq\cdots\succcurlyeq k_t$, along with 1, form a basis of $U(\mathfrak{gsv}[G,\alpha])$.

Let $c,h\in\mathbb{F}$, V_h be a 1-dimensional vector space over $\mathbb{F}$ spanned by v_h, i.e., $V_h=\mathbb{F}v_h$. View V_h as a $\mathfrak{gsv}[G,\alpha]_0$-module such that $L_0.v_h = hv_v$, $M_0.v_h = cv_h$. Then V_h is a $\mathfrak{B} = \mathfrak{gsv}[G,\alpha]_+ \oplus \mathfrak{gsv}[G,\alpha]_0$-module by setting $\mathfrak{gsv}[G,\alpha]_+.V_h = 0$.

Definition 4.1. The induced module $V(c,h) = \mathrm{Ind}_{\mathfrak{B}}^{\mathfrak{gsv}[G,\alpha]} V_h = U(\mathfrak{gsv}[G,\alpha]) \otimes_{U(\mathfrak{B})} V_h$ is called the Verma module of $\mathfrak{gsv}[G,\alpha]$ with highest weight (c,h).

Let $U := U(\mathfrak{gsv}[G,\alpha])$. For any $c,h\in\mathbb{F}$, let $I(c,h)$ be the left ideal of U generated by the elements

$$\{L_u, M_u, Y_{\alpha+v} \mid u\in G_+,\ \alpha+v\in G_{1+}\} \cup \{L_0-h, M_0-c\},$$

where $G_+ = G\cap T_+$, $G_{1+} = G_1\cap T_+$. Then the Verma module with highest weight (c,h) for $\mathfrak{gsv}[G,\alpha]$ also can be defined as $V(c,h) := U/I(c,h)$.

By definition, we can easily get a basis of $V(c,h)$ consisting of all vectors of the form:

$$v_h,\quad L_{-i_1}\cdots L_{-i_r}M_{-j_1}\cdots M_{-j_s}Y_{\alpha-k_1}\cdots Y_{\alpha-k_t}v_h,$$

where

$$0\prec i_1\preccurlyeq\cdots\preccurlyeq i_r,\quad 0\prec j_1\preccurlyeq\cdots\preccurlyeq j_s,\quad \alpha\prec k_1\preccurlyeq\cdots\preccurlyeq k_t;\quad r,s,t\in\mathbb{N}.$$

Remark. One can see that M_0 acts as a scalar c on $V(c,h)$ since $\mathbb{F}M_0$ is the center of $\mathfrak{gsv}[G,\alpha]$. Next, we call a vector $v\in V(c,h)$ a weight vector with weight μ if $L_0v=\mu v$.

Lemma 4.2. *$V(c,h)$ is a weight module of $\mathfrak{gsv}[G,\alpha]$, and $V(c,h)=\bigoplus_{\mu\in h-T_+}V_\mu$, where $V_\mu=\mathrm{span}\{L_{-i_1}\cdots L_{-i_r}M_{-j_1}\cdots M_{-j_s}Y_{\alpha-k_1}\cdots Y_{\alpha-k_t}v_h \mid -\sum_{p=1}^r i_p-\sum_{p=1}^s j_p-\sum_{p=1}^t(\alpha-k_p)=h-\mu\}$ is the weight vector space with weight μ.*

Proof. It suffices to show that L_0 acts diagonally on the basis elements of $V(c,h)$. By the definition of v_h, $L_0v_h=hv_h$. Suppose $u\in V(c,h)$ such that $L_0u=au$. Then

$$L_0(L_{-i}u)=(a-i)L_{-i}u,\quad L_0(M_{-j}u)=(a-j)M_{-j}u,\quad L_0(Y_{\alpha-k}u)=(a+\alpha-k)Y_{\alpha-k}u.$$

Thus Lemma 4.2 holds. $\square$

We know from [4] that for the fixed total order "$\succcurlyeq$" of T, either "$\succcurlyeq$" is dense, i.e., $\forall x \in T_+$, the cardinality of $\{y \in T \mid 0 \prec y \prec x\}$ is infinite, or "$\succcurlyeq$" is discrete, i.e., there exists $a \in T$ such that the set $\{y \in T \mid 0 \prec y \prec a\}$ is empty.

For the generalized Virasoro algebra $Vir[G]$ studied in [4], the irreducibility of Verma module over $Vir[G]$ depends on whether the total order of G is dense or discrete (see Theorem 3.1 in [4]). With respect to Verma modules over generalized Witt algebras studied in [9], the irreducibility depends on the action of L_0 on the highest weight vector (see Theorem 3 in [10]). It is very interesting that the irreducibility of Verma modules over $\mathfrak{gsv}[G,\alpha]$ depends neither on the action of L_0 nor on whether the total order is dense or discrete. We point out that the irreducibility just depends on the action of the element M_0.

For $x \in V(c,h)$, we set

$$x = \sum_{\substack{i_1 \preccurlyeq \cdots \preccurlyeq i_r,\, j_1 \preccurlyeq \cdots \preccurlyeq j_s,\, k_1 \preccurlyeq \cdots \preccurlyeq k_t \\ i_1,\dots,i_r,\, j_1,\dots,j_s \in G_+,\, k_1-\alpha,\dots,k_t-\alpha \in G_{1+}}} a_{\underline{i},\underline{j},\underline{k}} L_{-i_1}\cdots L_{-i_r} M_{-j_1}\cdots M_{-j_s} Y_{\alpha-k_1}\cdots Y_{\alpha-k_t} v_h,$$

where $a_{\underline{i},\underline{j},\underline{k}} \in \mathbb{F}$, $\underline{i} = (i_1,\dots,i_r)$, $\underline{j} = (j_1,\dots,j_s)$, $\underline{k} = (\alpha-k_1,\dots,\alpha-k_t)$, and only finitely many $a_{\underline{i},\underline{j},\underline{k}} \neq 0$. We define

$$A_x := \bigl\{\underline{i} = (i_1,\dots,i_r) \bigm| a_{\underline{i},\underline{j},\underline{k}} \neq 0 \text{ for some } \underline{j},\underline{k}\bigr\}, \qquad l = \max\bigl\{r \bigm| \underline{i} = (i_1,\dots,i_r) \in A_x\bigr\},$$

where $l = 0$ if $A_x = \emptyset$. We also define l to be the length of the element x, and denote it by $len(x)$, i.e., $l = len(x)$.

For $r \in \mathbb{N}$, we set

$$V_r := \operatorname{span}_{\mathbb{F}}\bigl\{x \bigm| len(x) \leqslant r\bigr\}.$$

In what follows, we assume $V_r = 0$ if $r \leqslant -1$. One can check the following two lemmas by straightforward and easy computations.

Lemma 4.3.

(i)

$$\begin{aligned} &M_j L_{-i_1} L_{-i_2} \cdots L_{-i_r} M_{-j_1} \cdots M_{-j_s} Y_{\alpha-k_1} \cdots Y_{\alpha-k_t} v_h \\ &\quad \equiv -j\biggl(\sum_{1 \leqslant p \leqslant r} L_{-i_1} \cdots \hat{L}_{-i_p} \cdots L_{-i_r} M_{j-i_p}\biggr) M_{-j_1} \cdots M_{-j_s} Y_{\alpha-k_1} \cdots Y_{\alpha-k_t} v_h \pmod{V_{r-2}}, \end{aligned}$$

for any $r \in \mathbb{N}$; $j \in G_+$; $0 \prec i_1 \preccurlyeq \cdots \preccurlyeq i_r$; $0 \prec j_1 \preccurlyeq \cdots \preccurlyeq j_s$; $0 \prec k_1 - \alpha \preccurlyeq \cdots \preccurlyeq k_t - \alpha$, where $\hat{}$ means the corresponding element is deleted.

(ii)

$$L_j M_{-i_1} M_{-i_2} \cdots M_{-i_r} v_h = \biggl(\sum_{1 \leqslant p \leqslant r} (-i_p) M_{-i_1} \cdots \hat{M}_{-i_p} \cdots M_{-i_r} M_{j-i_p}\biggr) v_h,$$

for any $r \in \mathbb{N}$; $j, i_1, \dots, i_r \in G_+$. In particularly,

$$L_j M_{-i_1} M_{-i_2} \cdots M_{-i_r} v_h = 0, \quad \forall j \succ \max\{i_1,\dots,i_r\}.$$

Lemma 4.4.

(i) $Y_{-\alpha+j}Y_{\alpha-k_1}Y_{\alpha-k_2}\cdots Y_{\alpha-k_t}v_h=0$, $\forall j\succ k_t$, *where* $\alpha\prec k_1\preccurlyeq\cdots\preccurlyeq k_t$.
(ii) *If* $M_0.v_h=0$, *then* $Y_{-\alpha+j}Y_{\alpha-k_1}Y_{\alpha-k_2}\cdots Y_{\alpha-k_t}v_h=0$, $\forall j\succcurlyeq k_t$, *where* $\alpha\prec k_1\preccurlyeq\cdots\preccurlyeq k_t$.

Corollary 4.5. $M_jV_r\subseteq V_{r-1}$, *for any* $j\in G_+$.

Proof. It follows immediately from Lemma 4.3(i). $\square$

The following theorem is our main result in this section. We remark that there is different proof for Theorem 4.6(i), in the case $\mathfrak{sv}$, given by C. Roger and J. Unterberger by using Kac determinants (see arXiv version of [12]).

Theorem 4.6.

(i) *The Verma module* $V(c,h)$ *is an irreducible* $\mathfrak{gsv}[G,\alpha]$ *module if* $c\neq 0$.
(ii) *If* $c=0$, *then the Verma module* $V(0,h)$ *contains a unique maximal proper submodule* $N(0,h)$, *where* $N(0,h)$ *is generated by* $\{L_{-u}v_h, M_{-u}v_h, Y_{\alpha-v}v_h \mid u\in G_+,\ v-\alpha\in G_{1+}\}$ *if* $h=0$, *by* $\{M_{-u}v_h, Y_{\alpha-v}v_h \mid u\in G_+,\ v-\alpha\in G_{1+}\}$ *if* $h\neq 0$.

Proof. (i) Suppose $c\neq 0$. Let $u_0\neq 0$ be any given weight vector in $V(c,h)$. By Lemma 4.2 and the fact that a submodule of a weight module is a weight module, we need only to prove that $v_h\in U(\mathfrak{gsv}[G,\alpha])u_0$.

Claim I. *There exists a weight vector* $u\in U(\mathfrak{gsv}[G,\alpha])u_0$ *such that*

$$u=\sum_{\substack{j_1\preccurlyeq\cdots\preccurlyeq j_s;\ k_1\preccurlyeq\cdots\preccurlyeq k_t\\ j_1,\dots,j_s\in G_+;k_1-\alpha,\dots,k_t-\alpha\in G_{1+}}} a_{\underline{j},\underline{k}}M_{-j_1}\cdots M_{-j_s}Y_{-k_1+\alpha}\cdots Y_{-k_t+\alpha}v_h,$$

where $a_{\underline{j},\underline{k}}\in\mathbb{F}$ *and only finitely many* $a_{\underline{j},\underline{k}}\neq 0$, $\underline{j}=(j_1,\dots,j_s)$, $\underline{k}=(k_1,\dots,k_t)$.

In fact, suppose

$$u_0=\sum_{\substack{0\prec i_1\preccurlyeq\cdots\preccurlyeq i_r\\ 0\prec j_1\preccurlyeq\cdots\preccurlyeq j_s\\ \alpha\prec k_1\preccurlyeq\cdots\preccurlyeq k_t}} a_{\underline{i},\underline{j},\underline{k}}L_{-i_1}\cdots L_{-i_r}M_{-j_1}\cdots M_{-j_s}Y_{\alpha-k_1}\cdots Y_{\alpha-k_t}v_h$$
$$\equiv\sum_{\substack{0\prec i_1\preccurlyeq\cdots\preccurlyeq i_l\\ 0\prec j_1\preccurlyeq\cdots\preccurlyeq j_s\\ \alpha\prec k_1\preccurlyeq\cdots\preccurlyeq k_t}} a_{\underline{i},\underline{j},\underline{k}}L_{-i_1}\cdots L_{-i_l}M_{-j_1}\cdots M_{-j_s}Y_{\alpha-k_1}\cdots Y_{\alpha-k_t}v_h \pmod{V_{l-1}},$$

where $l=len(u_0)$, $\underline{i}=(i_1,\dots,i_r)$, $\underline{j}$, $\underline{k}$ as above, $a_{\underline{i},\underline{j},\underline{k}}\in\mathbb{F}$.

If $l=0$, there is nothing to prove. Now suppose $len(u_0)=l\geqslant 1$, and let

$$i_l^{(0)}:=\max\bigl\{i_l\bigm|(i_1,\dots,i_l)\in A_{u_0}\bigr\},$$

where $A_{u_0}:=\{\underline{i}\mid\underline{i}=(i_1,\dots,i_l),\ a_{\underline{i},\underline{j},\underline{k}}\neq 0$ for some $\underline{j},\underline{k}\}$, then by using Lemma 4.3(i) and Corollary 4.4 we can deduce that

$$u_1 = M_{i_l^{(0)}} u_0 = \sum a_{\underline{i},\underline{j},\underline{k}} M_{i_l^{(0)}} L_{-i_1} \cdots L_{-i_r} M_{-j_1} \cdots M_{-j_s} Y_{\alpha-k_1} \cdots Y_{\alpha-k_t} v_h$$
$$\equiv \sum a_{\underline{i}^{(1)},\underline{j}^{(1)},\underline{k}^{(1)}} L_{-i_1^{(1)}} \cdots L_{-i_{l-1}^{(1)}} M_{-j_1^{(1)}} \cdots M_{-j_{s'}^{(1)}} Y_{\alpha-k_1^{(1)}} \cdots Y_{\alpha-k_{t'}^{(1)}} v_h \pmod{V_{l-2}}.$$

It is clear that $u_1 \neq 0$ (since $c \neq 0$) and $len(u_1) = l-1$.

Repeating the process and defining u_s recursively for $s = 2, \dots, l$, one obtains the claim.

Claim II. *There exists a weight vector $w \in U(\mathfrak{gsv}[G,\alpha])u_0$ such that w takes the following form*

$$w = \sum_{s \geqslant 0,\, j_1 \preccurlyeq \cdots \preccurlyeq j_s;\, j_1, \dots, j_s \in G_+} a_{\underline{j}} M_{-j_1} \cdots M_{-j_s} v_h.$$

In fact, by Claim I, we know that there is a weight vector $u \in U(\mathfrak{gsv}[G,\alpha])u_0$ such that

$$u = \sum_{\substack{s,t \geqslant 0,\, j_1 \preccurlyeq \cdots \preccurlyeq j_s \\ k_1 \preccurlyeq \cdots \preccurlyeq k_t;\, j_1, \dots, j_s \in G_+ \\ k_1-\alpha, \dots, k_t-\alpha \in G_{1+}}} a_{\underline{j},\underline{k}} M_{-j_1} \cdots M_{-j_s} Y_{-k_1+\alpha} \cdots Y_{-k_t+\alpha} v_h.$$

Set

$$B := \left\{ \underline{k} = (k_1, k_2, \dots, k_t) \,\middle|\, a_{\underline{j},\underline{k}} \neq 0 \text{ for some } \underline{j} \right\}, \quad k^{(0)} = \max\{k_t \mid \underline{k} \in B\}.$$

Then by Lemma 4.4, we have

$$w_1 = Y_{-(\alpha-k^{(0)})} u = \sum a_{\underline{j},\underline{k}^{(1)}} M_{-j_1} \cdots M_{-j_s} Y_{\alpha-k_1^{(1)}} \cdots Y_{\alpha-k_{t-1}^{(1)}} (M_0 v_h).$$

Note that $w_1 \neq 0$ and $w_1 \in U(\mathfrak{gsv}[G,\alpha])u_0$ is a weight vector. One repeats the process to get Claim II.

From Claim II we know that there exists a weight vector $w \in U(\mathfrak{gsv}[G,\alpha])u_0$ such that it has the following form

$$w = \sum_{s \geqslant 0,\, 0 \prec j_1 \preccurlyeq \cdots \preccurlyeq j_s} a_{\underline{j}} M_{-j_1} \cdots M_{-j_s} v_h.$$

We define

$$length(w) = \max\left\{ s \,\middle|\, \underline{j} = (j_1, \dots, j_s),\ a_{\underline{j}} \neq 0 \right\}.$$

If $length(w) = 0$, then $v_h \in U(\mathfrak{gsv}[G,\alpha])u_0$ and (i) holds. Now suppose $length(w) > 0$. Denote $j^{(0)} = \max\{j_s \mid \underline{j} = (j_1, \dots, j_s),\ a_{\underline{j}} \neq 0\}$. By applying $L_{j^{(0)}}$ to w and using Lemma 4.3(ii), we have

$$0 \neq w_1 = L_{j^{(0)}} w = \sum_{j_1^{(1)} \preccurlyeq \cdots \preccurlyeq j_s^{(1)};\, j_1^{(1)}, \dots, j_s^{(1)} \in G_+} a_{\underline{j}}^{(1)} M_{-j_1^{(1)}} \cdots M_{-j_s^{(1)}} M_0 v_h.$$

It is clear that

$$length(w_1) < length(w).$$

Repeating the process, we obtain

$$0 \neq w_s = aM_0 v_h = ac v_h \in U\big(\mathfrak{gsv}[G,\alpha]\big)u_0$$

for some $0 \neq a \in \mathbb{F}$. So $v_h \in U(\mathfrak{gsv}[G,\alpha])u_0$ and $V(c,h)$ is irreducible.

(ii) If $c=0$, $h=0$, by the definition of $N(0,0)$, one knows that all the basis elements of $V(0,0)$ except v_h are clearly in $N(0,0)$. It suffices to show that $v_h \notin N(0,0)$. For any weight vector $v \in N(0,0)$, suppose the weight of v is μ. For any basis element $L_{i_1}\cdots L_{i_r}M_{j_1}\cdots M_{j_s}Y_{\alpha+k_1}\cdots Y_{\alpha+k_t}$ of $U(\mathfrak{gsv}[G,\alpha])$ such that $\sum_{p=1}^r i_p + \sum_{p=1}^s j_p + \sum_{p=1}^t (\alpha+k_p) = -\mu$, we have

$$L_{i_1}\cdots L_{i_r}M_{j_1}\cdots M_{j_s}Y_{\alpha+k_1}\cdots Y_{\alpha+k_t} v = aL_0 v_h + bM_0 v_h = 0$$

for some $a,b \in \mathbb{F}$. This implies that $v_h \notin N(0,0)$.

If $c=0$, $h \neq 0$, similarly as above, we can see that $U(L_-)v_h \notin N(0,h)$, where $L_- = \bigoplus_{u \prec 0} \mathbb{F}L_u$. This means that $N(0,h)$ is a proper submodule of $V(0,h)$. Suppose V is any submodule of $V(0,h)$ such that $V \supsetneq N(0,h)$, then there exist $i_1,\dots,i_r \in G_+, r \in \mathbb{N}$ such that $L_{-i_1}\cdots L_{-i_r}v_h \in V$. If $r=0$, then $v_h \in V$ and $V = V(0,h)$. Suppose $r \geqslant 1$. We denote $i = i_1+i_2+\cdots+i_r$, then

$$L_i L_{-i_1}\cdots L_{-i_r} v_h = (-1)^r (i+i_1)(i-i_1+i_2)\cdots(i-i_1-i_2-\cdots-i_{r-1}+i_r)h v_h \in V.$$

Since $(-1)^r(i+i_1)(i-i_1+i_2)\cdots(i-i_1-i_2-\cdots-i_{r-1}+i_r)h \neq 0$ we have $v_h \in V$ and $V = V(0,h)$. So $N(0,h)$ is the unique maximal proper submodule of $V(0,h)$. $\square$

Remark. $V(0,0)/N(0,0) \simeq \mathbb{F}$ is a trivial module of $\mathfrak{gsv}[G,\alpha]$. $V(0,h)/N(0,h) \simeq U(L_-)$ as vector space.

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